THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 8

When we talk about connectedness, it is convenient to use the terms '*separation*'. A *separation* of a topological space X consists of two open sets U and V such that $U \cup V = X$ and $U \cap V = \emptyset$. Then a topological space X is connected if for any separation $\{U, V\}$ of X, we have $(U = X, V = \emptyset)$ or $(U = \emptyset, V = X)$.

1. Suppose X is finite. Let $X = \{x_1, x_2, \dots, x_n\}$. Since X is T_1, one can show that X is discrete (exercise). In particular, the set $\{x_1\}$ is both open and closed. By connectedness, we have $X = \{x_1\}$.

2. Let $\{U, V\}$ be a separation of X. We want to show that $\{p(U), p(V)\}$ is a separation of Y. To do so, we need to check:

   (i) $p(U)$ and $p(V)$ are open;

   (ii) $p(U) \cap p(V) = \emptyset$;

   (iii) $p(U) \cup p(V) = Y$

   (iii) is trivial since p is surjective.

   To show (ii), suppose there exists y such that $y \in p(U) \cap p(V)$. Then $\{U \cap p^{-1}(\{y\}), V \cap p^{-1}(\{y\})\}$ is a separation of $p^{-1}(\{y\})$. By connectedness, WLOG we assume that $V \cap p^{-1}(\{y\}) = \emptyset$. This implies $y \notin p(V)$, contradiction.

   To show (i), note that $U \subset p^{-1}(p(U))$. Suppose there exists $v \in V$ such that $v \in p^{-1}(p(U))$. Then $p(v) \in p(U)$, contradicting with (ii). Hence $U = p^{-1}(p(U))$ and similarly $V = p^{-1}(p(V))$. Hence $p(U)$ and $p(V)$ are open.

   As a result, $\{p(U), p(V)\}$ is a separation of Y. By connectedness of Y, WLOG we have $p(V) = \emptyset$. Hence $V = \emptyset$.

3. (a) For any $a \in A^c$ and $b \in B^c$, we define $U_{(a,b)} = (X \times \{b\}) \cup (\{a\} \times Y)$. Since $X \times \{b\} \cong X$ and $\{a\} \times Y \cong Y$, they are both connected. Furthermore, $(a, b) \in (X \times \{b\}) \cap (\{a\} \times Y)$. Hence $U_{(a,b)}$ is connected.

   Note that for any $a, a' \in A^c$, $b, b' \in B^c$, we have $(a, b') \in U_{(a,b)} \cap U_{(a',b')}$. Moreover, we have $(X \times Y)\backslash(A \times B) = \bigcup_{a \in A^c, b \in B^c} U_{(a,b)}$. Therefore, $(X \times Y)\backslash(A \times B)$ is connected.

   (b) * Here we only do the case where $n = 2$. Suppose there exists a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}$. Let $f(a, b) = 0$ for some $a, b \in \mathbb{R}$. Then the restriction of f from $\mathbb{R}^2\backslash(a, b)$ to $\mathbb{R}\backslash\{0\}$ is also a homeomorphism. Furthermore, since $\{a\}$ and $\{b\}$ are two proper subsets of $\mathbb{R}$, by a) we know that $\mathbb{R}^2\backslash(a, b)$ is connected. However, $\mathbb{R}\backslash\{0\}$ is not connected because of the fact that $\mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$. This leads to contradiction. Hence $\mathbb{R}^2$ and $\mathbb{R}$ are not homeomorphic.